J. DIFFERENTIAL GEOMETRY
16 (1981) 483–492

FAMILIES OF PERIODIC ORBITS: LOCAL CONTINUABILITY DOES NOT IMPLY GLOBAL CONTINUABILITY

KATHLEEN T. ALLIGOOD, JOHN MALLET-PARET &
JAMES A. YORKE

1. Introduction

For fixed points of zeroes of a map depending on a parameter, local continuability is closely related to global continuability. Let $F: \mathbf{R} \times \mathbf{R}^n \to \mathbf{R}^n$ be a C^1 function depending on a scalar parameter α. If $F(\alpha_0, x_0) = 0$, and $D_{(\alpha, x)}F(\alpha_0, x_0)$ has full rank, then the zero (α_0, x_0) is locally continuable in the sense that a path of zeroes extends from it through a neighborhood of (α_0, x_0). The global behavior of a connected component C of zeroes through (α_0, x_0) can also be described. We have two possibilities:

(a) $C - \{(\alpha_0, x_0)\}$ is connected; or

(b) both components of $C - \{(\alpha_0, x_0)\}$ are unbounded in (α, x)-space.

It is reasonable to say that the set of zeroes through (α_0, x_0) is *globally continuable* whenever C satisfies (a) or (b). The fact that these are the only possibilities is easily seen in the generic case (where $D_{(\alpha, x)}F(\alpha, x)$ has full rank whenever $F(\alpha, x) = 0$); it has also been shown to be true in the nongeneric case, assuming only that $D_X F(\alpha_0, x_0)$ is nonsingular [1]. Hence the conditions for local continuability in fact imply global continuability.

For solutions of a differential equation $dx/dt = F(\alpha, x)$, (again depending on a parameter α), we can relate the behavior of periodic orbits to that of fixed points. Each point on a periodic orbit is a fixed point of the Poincaré return map T (to be defined later) associated with the orbit at that point. (In the following, orbit will always mean periodic orbit.) Such an orbit is locally

Communicated by R. Bott, March 9, 1981. This research was supported in part by the National Science Foundation under Grants MCS-7818858 and MCS-7818221, by the Army Research Office under Grant AROD DA-ARO-D-31-124-78-G130, and by a College of Charleston Research Grant.

continuable if and only if each point on it is a locally continuable fixed point of T. The Poincaré index of the orbit is the Brouwer fixed point index of T. If the index is nonzero, then the orbit is locally continuable. In particular, the Poincaré index is nonzero if $D_x(T - \mathrm{id}_x)$ is nonsingular.

A generic family of periodic orbits exhibits global behavior analogous to that of fixed points—provided the initial orbit is locally continuable and is not a Möbius-type orbit. (Loosely speaking, a Möbius orbit is one whose unstable manifold is nonorientable.) The generic class considered here is discussed in [2]. For a connected component C of such a family and an orbit p_0 on C at $\alpha = \alpha_0$, at least one of the following must hold (see [2] and [4]):

(a) $C - \{(\alpha_0, p_0(t)): t \geqslant 0\}$ is connected; or each component of $C - \{(\alpha_0, p_0)\}$ either is

(b1) unbounded in (α, x)-space, or

(b2) has unbounded periods; or

(c) there is a generalized Hopf bifurcation, i.e., the diameter of the orbits goes to zero as the family approaches a stationary solution.

Any family of periodic orbits which satisfies one or more of the above conditions could be said to be *globally continuable.*

The question remained: are these the only possibilities for an orbit which is locally continuable, that is, for which the Poincaré index is nonzero? The objective of this paper is to show that the answer is no. We present an example of a differential equation $dx/dt = F(\alpha, x)$ and a particular (necessarily Möbius) orbit γ which has a nonzero Poincaré index, but which is not globally continuable. The orbit γ is contained in a family C of orbits such that one component of $C \setminus \gamma$ is bounded (with bounded periods), and the diameters of all orbits in C are strictly positive. (See [2] for the definition of an "orbit index" for which the index of γ is zero.)

We construct the example as follows. Let $f: \mathbf{R}^4 \to \mathbf{R}^4$ be a C^1 function such that $\dot{x} = f(x)$ has a Möbius orbit solution γ. We define a homotopy $f_\alpha = F(\alpha, \cdot)$ of f such that

(1) γ is contained in a family γ_α of Möbius orbits for α near α_0;

(2) a second family $\Gamma_{2,\alpha}$ or orbits (with approximately twice the period) bifurcates from Γ_α at $\alpha = \alpha_1$; and

(3) the family $\Gamma_{1,\alpha}$ (the low-period continuation of Γ_α for $\alpha > \alpha_1$) and the family $\Gamma_{2,\alpha}$ coalesce and annihilate each other at $\alpha = \alpha_2$.

The only orbits contained wholly within some ε-neighborhood of C are those in the families described. We should also note that this example persists under small C^1 perturbations and can be made real analytic.

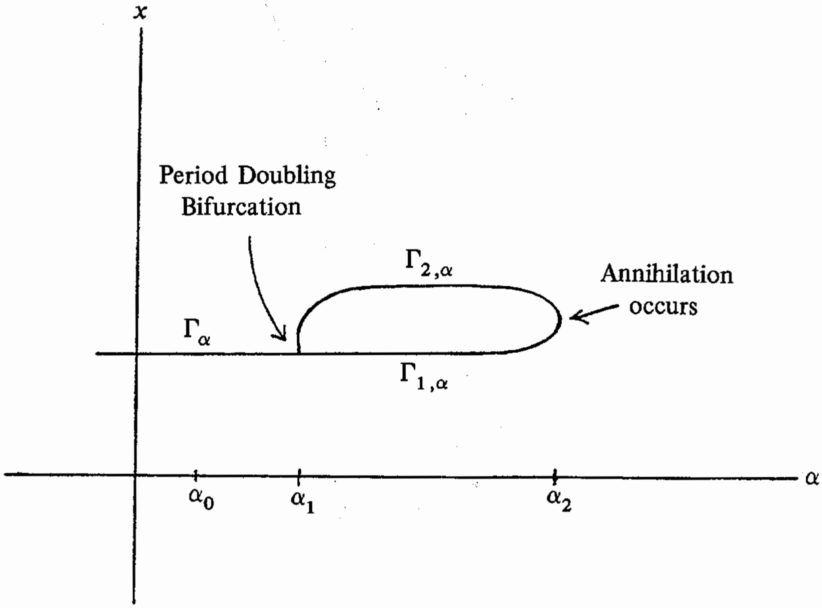


Fig. 1.1

A schematic diagram of the example is shown, in which each point on the 1-dimensional branched curve C represents a periodic orbit. For $\alpha < \alpha_1$, each orbit in the family Γ_α is a Möbius orbit. After the bifurcation at $\alpha = \alpha_1$, the orbits on the upper branch of C—the family $\Gamma_{2,\alpha}$—are all hyperbolic; the orbits on the lower branch $\Gamma_{1,\alpha}$ are attractors.

2. A globally noncontinuable example

Given a differential equation $dx/dt = F(\alpha, x)$, $F: \mathbf{R} \times \mathbf{R}^n \to \mathbf{R}^n$, the main tool for analyzing a periodic solution γ of F is the Poincaré map. Let (α_0, x_0) be a point on γ, and let D be an n-dimensional disk perpendicular to $(0, F(\alpha_0, x_0))$ at (α_0, x_0). The Poincaré map T is defined for (α_1, x_1) in D sufficiently close to (α_0, x_0) as follows: let $T(\alpha_1, x_1)$ be the x-coordinate of the point where the trajectory through (α_1, x_1) next hits D. (The α coordinate is α_1.) We say μ is a *multiplier* of γ if it is an eigenvalue of the $(n-1) \times (n-1)$ matrix of partial derivatives $D_x T(\alpha_0, x_0)$. An orbit with an odd number of multipliers (counted with multiplicities), in $(-\infty, -1)$ is called a *Möbius orbit.*

We begin with a differential equation

$$\frac{dx}{dt} = f(x) = F(\alpha_0, x),$$

where $f: \mathbf{R}^4 \to \mathbf{R}^4$ is infinitely differentiable and has a hyperbolic Möbius orbit γ as solution. γ has one multiplier $\mu_1 < -1$, and two multpliers μ_i, $i = 2, 3$, such that $-1 < \mu_2 < 0 < \mu_3 < 1$. (I.e., the orbit is unstable on an invariant Möbius band M, and M in turn is an attractor in $\mathbf{R}^4$. See Fig. 2.1.) For an orbit with no multipliers on the unit circle, such as γ, the Poincaré fixed point index of T is $(-1)^{\sigma^+}$, where σ^+ is the number of multipliers (counted with multiplicites) in $(1, \infty)$. Since the index is nonzero, we know that the fixed point x_0 (and hence the orbit γ) has a local continuation.

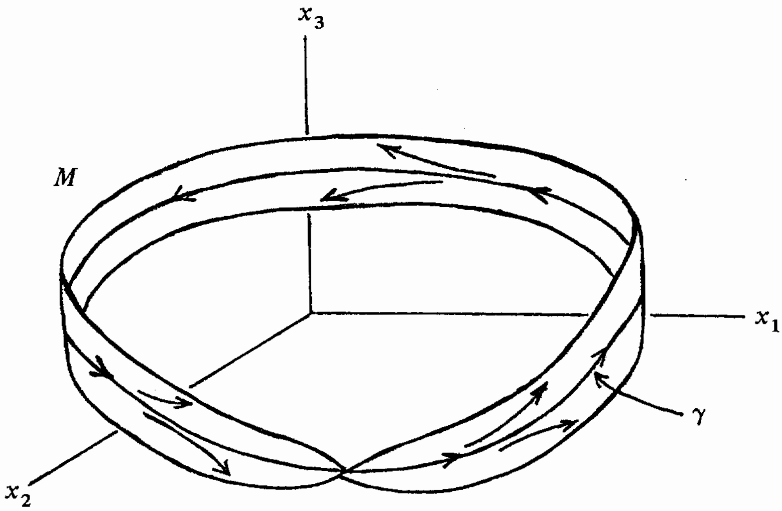


Fig. 2.1

The periodic orbit γ is shown in $\mathbf{R}^3 \times \{0\}$. The unstable manifold of γ is a Möbius band M. The Möbius band is an attractor in $\mathbf{R}^4$.

The example will be described in four main steps:

Step 1. *A period doubling bifurcation.* We perturb $\dot{x} = f(x)$ so that μ_1 crosses -1, resulting in a period doubling bifurcation from Γ_α. Let $\Gamma_{1,\alpha}$ be the continuation of the family Γ_α through low-period orbits, and let $\Gamma_{2,\alpha}$ be the family of double-period orbits. We will denote by γ_1 (respectively, γ_2) a single orbit on the family $\Gamma_{1,\alpha}$ (respectively, $\Gamma_{2,\alpha}$). Notice that γ_1 is an attractor; and γ_2, which is unstable on the Möbius band M, has an orientable neighborhood in M (Fig. 2.2).

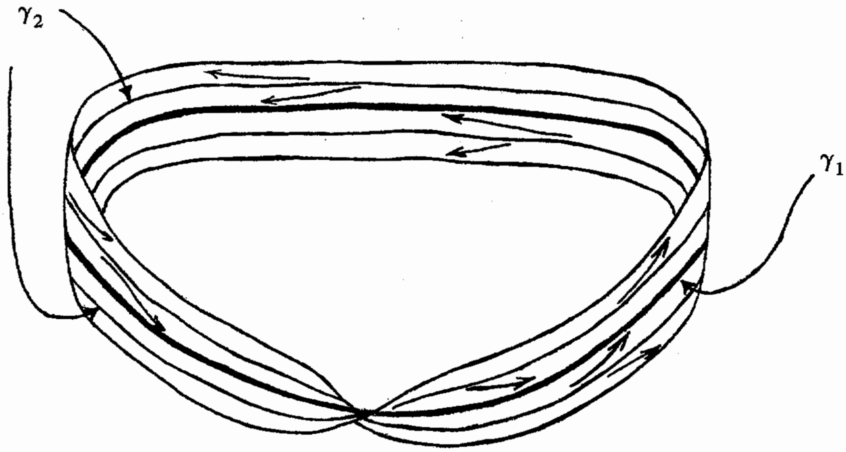


Fig. 2.2

The unstable family of orbits Γ_α has undergone a period doubling bifurcation. Orbits γ_1 on the stable family $\Gamma_{1,\alpha}$ and γ_2 on the unstable family $\Gamma_{2,\alpha}$ are shown. The period of γ_2 is approximately twice that of γ_1. The Möbius band M remains an attractor in $\mathbf{R}^4$.

Step 2. *Unlinking the orbits.* In Fig. 2.3 we see that γ_1 and γ_2, as subsets of the Möbius band, are linked in $\mathbf{R}^3$.

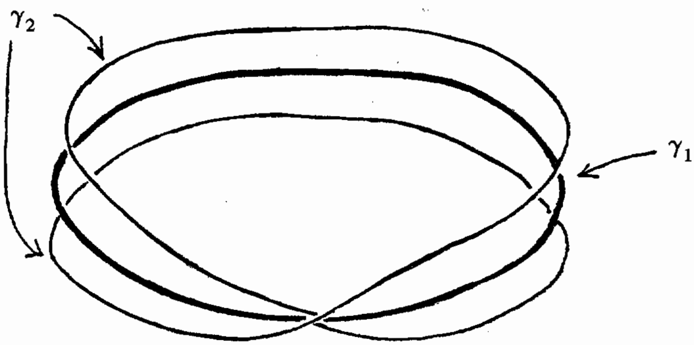


Fig. 2.3

The orbits γ_1 and γ_2 are linked as subsets of the Möbius band in $\mathbf{R}^3 \times \{0\}$.

We shall proceed with the deformation of $\dot{x} = f(x)$ by indicating how neighborhoods of the orbits move continuously through $\mathbf{R}^4$. Let N_1 and N_2 be closed disjoint tubular neighborhoods of γ_1 and γ_2 respectively in $\mathbf{R}^4$. Technically, this continuous motion is an isotopy $G\colon I \times (N_1 \cup N_2) \to \mathbf{R}^4$. (An isotopy is a homotopy of embeddings $g_\alpha\colon N_1 \cup N_2 \to \mathbf{R}^4$.) We can extend G to an ambient isotopy $H\colon I \times \mathbf{R}^4 \to \mathbf{R}^4$, (an isotopy of diffeomorphisms). Let $\dot{x} = F(a_*, x)$ be the differential equation described above with periodic solutions γ_1 and γ_2. Define $F\colon I \times \mathbf{R}^4 \to \mathbf{R}^4$ as follows: if y is the point such that $H(\alpha, y) = x$, let $F(\alpha, x) = D_x H(\alpha, y) F(\alpha_*, y)$. In other words, move the solution curves via the function H; then calculate the tangents to these curves to get the vector field.

Assume that the Möbius band M lies in $\mathbf{R}^3 \times \{0\}$; i.e., $p_4(M) = 0$, where $p_i\colon \mathbf{R}^4 \to \mathbf{R}$ is the projection on the ith factor, $i = 1, \cdots, 4$. We will use the 4th coordinate to unlink γ_1 and γ_2 by homotoping γ_1 away from $x_4 = 0$ so that $p_4(N_1) \cap p_4(N_2)$ is empty. For ease of conceptualization, we can now also unlink γ_1 and γ_2 in $\mathbf{R}^3$, (e.g., let $p_3(N_1) \cap p_3(N_2)$ be empty, as Fig. 2.4 represents). Let $\tilde{N}_1 = N_1 \cap \mathbf{R}^3$.

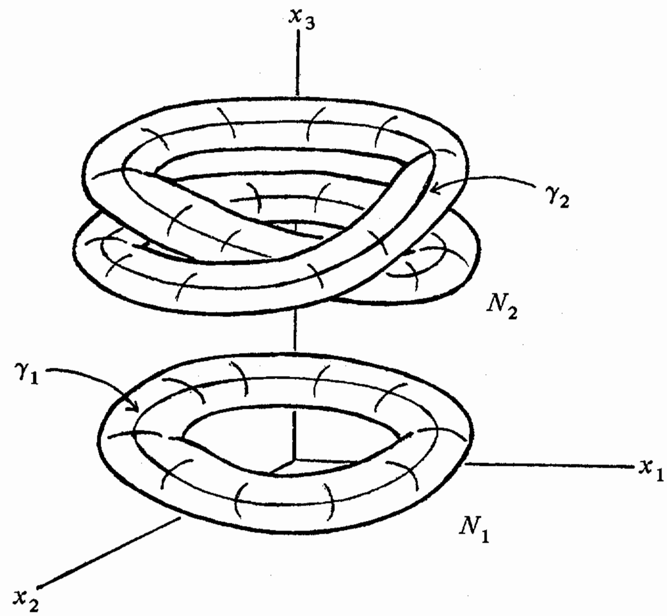
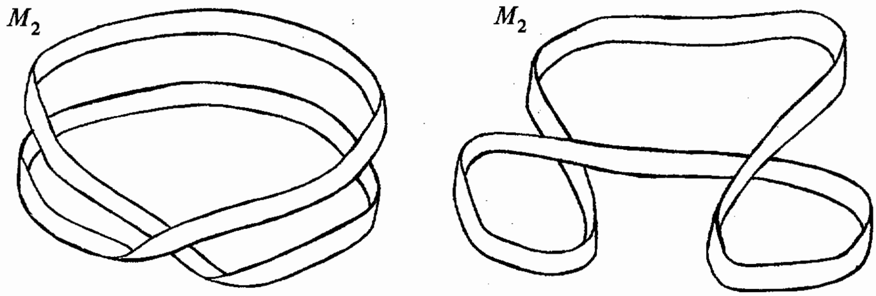


Fig. 2.4

Tubular neighborhoods $\tilde{N}_1$ and $\tilde{N}_2$ of the orbits γ_1 and γ_2 respectively are shown. The neighborhoods are now disjoint in $\mathbf{R}^3$, after being unlinked in $\mathbf{R}^4$.

Step 3. *Untwisting the flow about* γ_1. Since $\tilde{N}_1 - \gamma_1$ is homeomorphic to $S^1 \times S^1 \times \mathring{I}$, we can describe points in this open set in terms of two angles, θ_1 and θ_2, and the distance ρ from γ_1. Notice that with these coordinates, $d\rho/dt < 0$ (since γ_1 is an attractor), and $d\theta_2/dt$, the rate of twist of trajectories around γ_1, is nonzero. Let W be a closed tubular neighborhood of γ_1 properly contained in $\tilde{N}_1$. In order to eliminate the "Möbius" twist in W, we homotop $d\theta_2/dt$ to 0, keeping dx/dt unchanged outside $\tilde{N}_1$. Relabel W as N_1. Now γ_1 is an attractor with multipliers $0 < \mu_i < 1$ for $i = 1, 2, 3$.

Step 4. *Annihilation of the two families.* If we look at what has happened to the original two-dimensional neighborhoods $N_1 \cap M = M_1$ and $N_2 \cap M = M_2$, we see that γ_1 is unstable and γ_2 is stable in the orientable neighborhoods M_1 and M_2, respectively. In Fig. 2.5(a) we see M_2 as a subset of M. Fig. 2.5(b) shows the same M_2 in which the doubly-twisted band has been isotoped (in $\mathbf{R}^3$) to an "interwoven" one without twists.

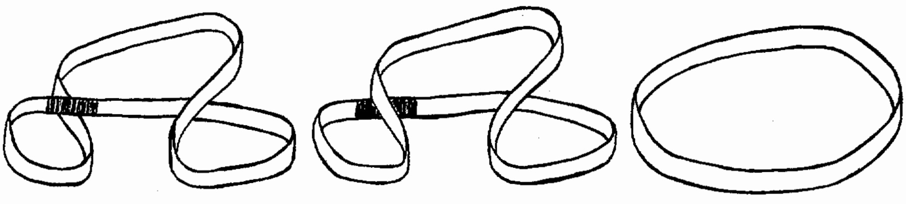


(a) (b)

Fig. 2.5

Two isotopic representations of the 2-dimensional neighborhoods M_2 of γ_2 are shown. The drawing in (a) depicts M_2 as a subset of the Möbius band.

The successive drawings in Fig. 2.6 represent an isotopy of M_2 in $\mathbf{R}^4$ which eliminates the "weave" in M_2. The crossing indicated in Fig. 2.6(b) requires a deformation of the shaded portion in the 4th coordinate as in the earlier argument.



(a) (b) (c)

Fig. 2.6

The three successive drawings represent an isotopy in $\mathbf{R}^4$ of the neighborhood M_2. The transition from (a) to (b) requires a deformation of the shaded portion in the 4th dimension in order to avoid self-intersection.

Before continuing with the final steps of the deformation, we claim that the families $\Gamma_{1,\alpha}$ and $\Gamma_{2,\alpha}$ have thus far been isolated from other periodic orbits which might have been created throughout the homotopy of $\dot{x} = f(x)$. Orbits in the family $\Gamma_{1,\alpha}$ are attractors and thus can be the only orbits inside a neighborhood of $\Gamma_{1,\alpha}$. Since the orbits in $\Gamma_{2,\alpha}$ are all hyperbolic, they also are isolated within a neighborhood of $\Gamma_{2,\alpha}$, as the following argument shows.

Suppose there exists a sequence of orbits $(\eta_i)_{i\in\mathbf{N}}$, not contained in $\Gamma_{2,\alpha}$, converging to an orbit γ_2 in $\Gamma_{2,\alpha}$. Let (α_0, x_0) be a point on γ_2, T_0 be the Poincaré map for γ_2 at (α_0, x_0), and A_0 be the matrix $D_xT_0(\alpha_0, x_0)$. Assume further that (α_i, x_i) is a point on η_i, and that T_i is the Poincaré map for η_i, at (α_i, x_i), and A_i is the matrix $D_xT_i(\alpha_i, x_i)$. Then there will be a sequence of points $(\alpha_i, x_i)_{i\in\mathbf{N}}$ converging to (α_0, x_0) such that $T_i^m(\alpha_i, x_i) = (\alpha_i, x_i)$ for some $m \geqslant 1$. If $m = 1$, two sequences of fixed points of the T_i's converge to (α_0, x_0). But this contradicts the fact that $I - A_0$ is an isomorphism. For $m > 1$, we refer to the following theorem.

Theorem [3]. *Let $T: \mathbf{R}^n \to \mathbf{R}^n$ be C^1, and let 0 be an isolated fixed point of each iterate T^k, although the neighborhood of isolation may depend on k. Let $m > 1$ be an integer. Let $\varepsilon > 0$ and let*

$$B(\varepsilon) = \{x \in \mathbf{R}^n \,|\, |x| \leqslant \varepsilon\}.$$

If $S: \mathbf{R}^n \to \mathbf{R}^n$ is sufficiently near T in the C^1 norm on this disk, that is, if $|T - S|_{C^1(B(\varepsilon))} \ll 1$, then a necessary condition for there to exist $x \in B(\varepsilon)$ with x, $S(x), \cdots, S^{m-1}(x)$ distinct but $S^m(x) = x$ is that there exist $y \in \mathbf{R}^n$ with $y, Ay, \cdots, A^{m-1}y$ distinct but $A^m(y) = y$.

For i sufficiently large, T_i satisfies the conditions on S. Hence if (α_i, x_i), $T_i(\alpha_i, x_i), \cdots, T_i^{m-1}(\alpha_i, x_i)$ are distinct, and $T_i^m(\alpha_i, x_i) = (\alpha_i, x_i)$, then there will exist a point y in $\mathbf{R}^3$ such that $y, Ay, \cdots, A^{m-1}y$ are distinct, and $A^m y = y$. But A^j, for all $j \in \mathbf{N}$, has only one fixed point, namely 0. Thus the claim is verified.

Proceeding with the deformation, we now stretch N_1 so that the length of γ_1 is equal to that of γ_2, and move γ_1 back into $\mathbf{R}^3 \times \{0\}$, (see Fig. 2.7(a)). As γ_1 and γ_2 are homotoped together, we let dx_3/dt go to zero at points between the orbits in M_1 and M_2. Of course, dx_3/dt must be kept nonzero at points in $M_1 - \gamma_1$ and $M_2 - \gamma_2$ so that no new periodic orbits are introduced (Fig. 2.7(b)). Finally, we have one orbit γ remaining (Fig. 2.7(c)) which disappears as dx_3/dt is homotoped from zero to a positive value for (some) points on γ (Fig. 2.7(d)).

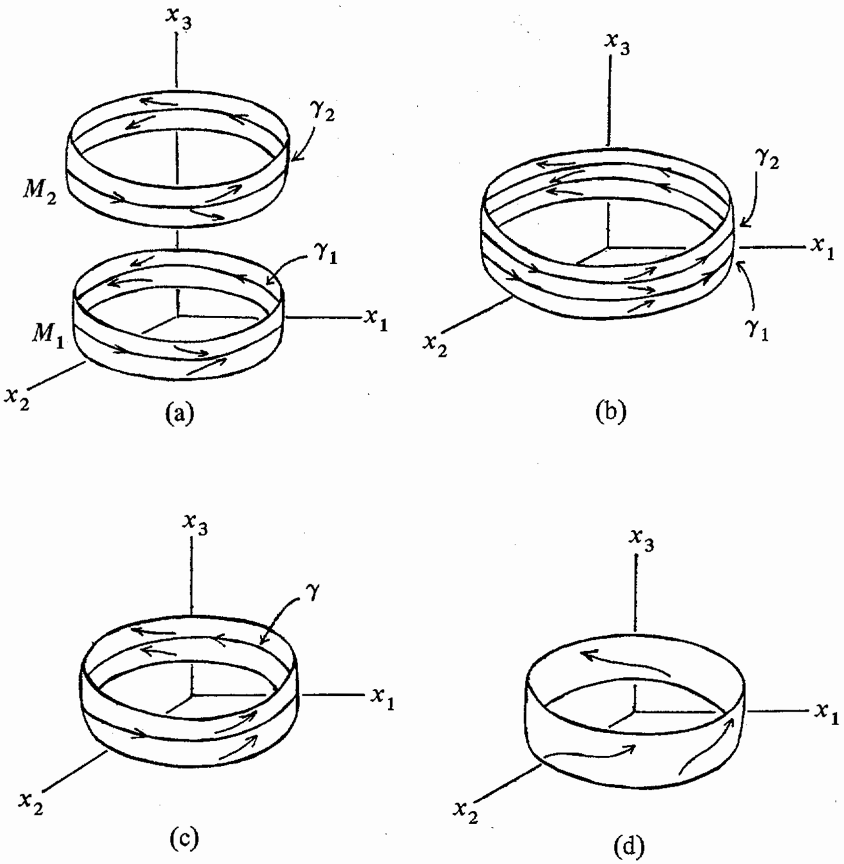


Fig. 2.7

The orbits γ_1 and γ_2 are homotoped together. As the drawing in (a) shows, dx_3/dt is negative for points on M_1 and M_2 between the orbits. In (b) both orbits are shown in the same neighborhood. As γ_1 and γ_2 are brought together from (b) to (c), dx_3/dt goes to zero at these points. Finally, from (c) to (d), dx_3/dt is homotoped from zero to some positive value for points on the (now single) orbit γ, and the orbit disappears.

References

[1] J. C. Alexander & J. A. Yorke, *Global bifurcation of periodic orbits*, Amer. J. Math. **100** (1978) 263–292.

[2] J. Mallet-Paret & J. A. Yorke, *Snakes: Oriented families of periodic orbits, their sources, sinks, and continuation*, J. Differential Equations, in press.

[3] ______, *A bifurcation invariant: degenerate periodic orbits treated as clusters of simple orbits*, preprint.

[4] ______, *Two types of Hopf bifurcaton points: sources and sinks of families of periodic orbits*, Proc. New York Academy of Sciences Meeting on Nonlinear Dynamics, 1979.

COLLEGE OF CHARLESTON
BROWN UNIVERSITY
UNIVERSITY OF MARYLAND